Intro GGT — Exercises 2

Exercise 1 (Rank)**.** We shall use the notation $[a, b] = aba^{-1}b^{-1}$.

1. Let G be a group. Let $[G, G]$ be the *subgroup generated* by the set $\{[g, h] \mid g, h \in G\}$. Show that $[G, G]$ is a normal subgroup of G.
2. Show that the quotient $G_{ab} := G/[G, G]$ is an abelian group.

   This group is called the **abelianization** of G.
3. Let S be a finite set with $|S| = n$, and let $F(S)$ be the free group on S. Prove that $F(S)_{ab} \cong \mathbb{Z}^n$.
4. Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^m$ be a group morphism. Construct a linear map $\widetilde{\varphi} : \mathbb{R}^n \to \mathbb{R}^m$ that extends φ via canonical injections. Show that if $\widetilde{\varphi}$ is not sujective then φ is not surjective.
5. Deduce that $\mathbb{Z}^n \cong \mathbb{Z}^m$ if and only if $n = m$.
6. Deduce that $F(S) \cong F(S')$ if and only if $|S| = |S'|$.
7. Deduce that if $S$and S are two free generating sets of a free group F, then $|S| = |S'|$.

   The cardinality $|S|$ is called the **rank** of F.

Exercise 2 (Baumslag-Solitar)**.** Consider the **Baumslag-Solitar group** defined by

$$\mathrm{BS}(m, n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle$$

where $m, n \in \mathbb{N}$.

1. Identify $\mathrm{BS}(m, 0)$, $\mathrm{BS}(0, n)$ and $\mathrm{BS}(1, 1)$.
2. Show that every element of $\mathrm{BS}(1, n)$ can be written in the form $b^{-j}a^k b^l$ for some $j, l \in \mathbb{N}$ and $k \in \mathbb{Z}$.
3. Describe the abelianization $\mathrm{BS}(m, n)_{ab}$.
4. Consider the following two matrices in $GL_2(\mathbb{R})$:

   $$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{n}{m} & 0 \\ 0 & 1 \end{pmatrix}.$$

   Show that there is an epimorphism from $\mathrm{BS}(m, n)$ to the subgroup of $GL_2(\mathbb{R})$ generated by A and B. Is it always an isomorphism?
5. Deduce that $\mathrm{BS}(m, n)$ is infinite and not cyclic. *(Hint: Suppose that it's cyclic, show that $A^p = B^q$ for some integers p and q)*
6. Consider the map

   $$\begin{array}{rccc} \varphi\colon & \mathrm{BS}(2,3) & \to & \mathrm{BS}(2,3) \\ & a & \mapsto & a^2 \\ & b & \mapsto & b \end{array}$$

   Show that φ is a well-defined epimorphism.
7. Show that $bab^{-1}aba^{-1}b^{-1}a^{-1}$ is a non-trivial element in $\ker(\varphi)$. Deduce that φ is not an isomorphism.